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Optimal Adaptive Integer Pulse Control of Stochastic Nonlinear Systems: Application to the Wolf-Moose Predator Prey System

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ABSTRACT
A numerical approach to optimal adaptive integer pulse control of stochastic nonlinear systems is presented. For most stochastic nonlinear systems, optimal adaptive control rules cannot be derived with analytical methods. A robust optimization algorithm is created. The complete nonlinear adaptively controlled stochastic system is simulated during 100 years, for 100 alternative sequences of stochastic disturbances, for every feasible integer combination of adaptive control rules. The optimal adaptive control rules that maximize the expected value of the objective function are selected as the optimal adaptive control rules. The method is very general and can easily be applied to most adaptive nonlinear stochastic control problems, from technology, management or other fields. The method is tested and applied to the wolf-moose predator prey system. The parameters of this stochastic nonlinear dynamical system have recently been estimated from empirical data from Isle Royale in Lake Superior, USA. The objective function is the expected total present value of all hunting net revenues and the environmental value of preserving the wolf population. The value of the wolf population is a strictly increasing and strictly concave function of the population level. Periodically, the region is visited and the population levels are determined. If the population levels, one for each species, exceed the optimal control limits, then the populations are reduced to the control limits, via hunting. Then, the system is left to develop until the next period. Optimal population control limits and objective function values are determined for alternative levels of the wolf population value function. The average optimal moose hunting level is a decreasing function of the wolf population value parameter and an increasing function of the level of risk in the predator prey system. The average optimal wolf population level is an increasing function of the wolf population value parameter and a decreasing function of the level of risk in the predator prey system.

KEYWORDS
Adaptive Optimization; Stochastic Nonlinear System; Predator Prey; Moose; Wolf

1. Introduction

Dynamical systems can be studied via differential and/or difference equations. Difference equations are relevant when we treat time as discrete. In continuous time, we use differential equations. A very large part of the theoretical system dynamics litera-

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ture focuses on differential equations. One important reason probably is that explicit solutions often can be obtained via analytical solutions. The field is classical and the ambition has often been to derive explicit solutions in continuous time. Often, discrete time and difference equations are however more suitable for the real problems at hand. Many of the classical dynamic analysis methods and problems are presented in the great books by [2], [1] and [14].
Often, the classical applications of dynamic system theory come from ecological problems. [8] was a pioneer who determined solutions to predator prey systems with analytical methods in continuous time. [4] and [6] contribute, with the help of empirical data and observations, to the applications and theories of such systems. Optimal control of dynamical systems, in particular under risk, is an area that has gained considerable interest. [3], [13] and [16] contain most of the theories and methods.
In recent years, optimal control problems have gained considerable attention in many application areas. As the level of detail increases, it is often found that the classical optimization approaches based on assumptions of differentiable functions, certainty and analytical solutions have to be modified. Relevant assumptions are needed in order to model relevant problems. [11] introduce the field of modeling and control of systems that simultaneously include discrete and continuous dynamics. They denote such systems hybdrid systems, or cyber-physical systems. In order to model and control such systems, several new methods may be needed.
One of the early applications of the hybrid system ideas is presented in [17]. They show that manufacturing processes can be considerably improved via integration of system state monitoring and alternative adaptive control methods. They also perform real experiments to show how a machine can become more than 50% more efficient via the proposed adaptive approach. In logistics, we find very interesting applications of nonlinear and discontinuous control. [10] determine optimal adaptive control rules for unmanned aerial vehicles (UAV) under the influence of uncertainties and several nonlinearities. [15] study and optimize adaptive control rules for heavy trucks that follow each other on roads, in aeras where the terrain is not always flat and where the roads are not always straight. In both of these articles, numerical studies and simulations are included that show the gains obtained via the suggested adaptive approaches. Nonlinear and discontinuous adaptive control rules have also been found to be optimal in DC motors, by [12], and in other kinds of motors, by [5]. Both of these studies handle several nonlinearities and stochastic disturbances. In both studies, analytical methods are combined with numerical methods, simulations and practical experiments.
This paper focuses on a numerical approach to optimal adaptive integer pulse control of stochastic nonlinear systems. One important reason is, that for most stochastic nonlinear systems, optimal adaptive control rules cannot be derived with analytical methods. In special cases, however, continuous time and stochastic processes may be combined with analytical derivations of optimal adaptive control rules. One such example of an optimization analysis in continuous time, where risk is present, is [7]. There, however, only one species is controlled, namely moose. Furthermore, there are no nonlinearities in the stochastic process.
In order to test the general optimization method, we will apply it to the wolf-moose predator prey system. The parameters of this stochastic nonlinear dynamical system have recently been estimated from empirical data from Isle Royale, an island in Lake Superior, USA. The empirical data, representing the time period 1959 to 2019, was obtained from [9]. Periodically, the region is visited and the population levels are determined. If the population levels, one for each species, exceed the optimal control limits, then the populations are reduced to the control limits, via hunting. Then, the system

is left to develop until the next period. Optimal population control limits and objective function values will be determined for alternative levels of the wolf population value function.

2. Materials and methods

A robust optimization algorithm is created. The complete nonlinear adaptively controlled stochastic system is simulated during 100 years, for 100 alternative sequences of stochastic disturbances, for every feasible integer combination of adaptive control rules. The optimal adaptive control rules that maximize the expected value of the objective function are selected as the optimal adaptive control rules. The method is very general and can easily be applied to most adaptive nonlinear stochastic control problems, from technology, management or other fields.
The objective function is the total present value of all hunting net revenues and the environmental value of preserving the wolf population. The value of the wolf population is a strictly increasing and strictly concave function of the population level. The objective function, π, is presented in equation (1). The rate of interest is r. u(.) and v(.) are the numbers of hunted and killed moose and wolf, respectively. M(.) and W(.) are the sizes of the moose and wolf populations. $P_{ARW}\psi$(W(.)) is the environmental value of preserving the wolf population. The value of the wolf population is a strictly increasing and strictly concave function of the population level. In the software, found in the appendix, ψ(W) is defined as LN(1+W). P_M and P_W are the net profits per killed moose and wolf, respectively. The parameter values used in the optimizations are:
P_M = \$ 1000, P_W = \$ 1000 and r = 3%. In the different Figures, the values of P_{ARW} are different. All details are found in the Appendix.

$$\pi = E\left(\sum_t e^{-rt}\left(P_M u(t,\bullet) + P_W v(t,\bullet) + P_{ARW}\Psi(W(t,\bullet))\right)\right) \tag{1}$$

The simultaneous difference equation system in general form, found in (2), describes how the populations of moose and wolf, develop over time. The difference equations contain stochastic components, ϵ_M and ϵ_W. These are described in more details in (7) and (9).

$$\begin{cases} \Delta M & = \phi\left(M, W, u, \varepsilon_M\right) \quad ,\forall t \\ \Delta W & = \varphi\left(M, W, v, \varepsilon_W\right) \quad ,\forall t \end{cases} \tag{2}$$

In equation (3) we see how the expected value of the objective function is estimated from N complete stochastic scenarios. In every stochastic scenario, n, the adaptive control functions are the same. However, the random number sequences, $\epsilon_M(t,n)$ and $\epsilon_W(t,n)$, are different for different n.

$$\pi = N^{-1}\sum_t \sum_{n=1}^{N} e^{-rt}\left(P_M u(t,n) + P_W v(t,n) + P_{ARW}\Psi(W(t,n))\right) \tag{3}$$

Furthermore, all scenarios have the same initial conditions. These are found in (4).

$$(M(0,n), W(0,n)) = (M_0, W_0) = (1200, 25) \quad ,\forall n \tag{4}$$

When the optimal control functions, u*(t, n) and v*(t, n) and are known and utilized, we may denote the conditional moose and wolf population values optimal. These are M*(.) and W*(.). In (5), we see how the optimized control functions and values can be used to calculate the optimal expected value π^*.

$$\pi^* = N^{-1} \sum_t \sum_n e^{-rt} \left(P_M u^*(t,n) + P_W v^*(t,n) + P_{ARW} \Psi(W^*(t,n)) \right) \tag{5}$$

The empirical data from Isle Royal was used to derive the equations (6) – (11) via multiple regression analysis. The time intervals, Δt, in these regressions, were one year.
In equation (6), we see how the change of the moose population is affected by the sizes of the moose population and the wolf population. The standard deviations of the parameter values are given below the estimated parameter values, in brackets. The number of observations was 60 and the multiple R value was 0.367. All parameters in equations (6) and (8) obtained P-values below 5%.

$$\begin{array}{c} \Delta M = 0.372M - 0.178 \times 10^{-3} M^2 - 6.593W + M\varepsilon_M \\ [0.135] \quad \left[0.0845 \times 10^{-3}\right] \quad [2.215] \end{array} \tag{6}$$

The stochastic residual of (6) is obtained this way: A normally distributed random number, defined in (7), is multiplied by the size of the population.

$$\varepsilon_M \sim N\left(\mu_M, \sigma_M^2\right), \quad \mu_M = 0, \sigma_M \approx 0.174 \tag{7}$$

In equation (8), we see how the change of the wolf population is affected by the sizes of the moose population and the wolf population. The standard deviations of the parameter values are given below the estimated parameter values, in brackets. The number of observations was 60 and the multiple R value was 0.272.

$$\begin{array}{c} \Delta W = -0.244W + 0.230 \times 10^{-3} MW + W\varepsilon_W \\ [0.117] \quad \left[0.107 \times 10^{-3}\right] \end{array} \tag{8}$$

The stochastic residual of (8) is obtained this way: A normally distributed random number, defined in (9), is multiplied by the size of the population. The correlation between the random numbers (7) and (9) is zero.

$$\varepsilon_W \sim N\left(\mu_W, \sigma_W^2\right), \quad \mu_W = 0, \sigma_W \approx 0.334 \tag{9}$$

The stochastic nonlinear system, without controls, is (10):

$$\begin{cases} \Delta M = 0.372M - 0.178 \times 10^{-3} M^2 - 6.593W + M\varepsilon_M \\ \Delta W = -0.244W + 0.230 \times 10^{-3} MW + W\varepsilon_W \end{cases} \tag{10}$$

When we introduce the controls, u and v, that are functions of different variables and parameters, we get (11):

$$\begin{cases} \Delta M &= 0.372M - 0.178 \times 10^{-3}M^2 - 6.593W - u(\bullet) + M\varepsilon_M \\ \Delta W &= -0.244W + 0.230 \times 10^{-3}MW - v(\bullet) + W\varepsilon_W \end{cases} \tag{11}$$

In several cases, it is interesting to see how the amount of risk in the difference equations of moose and wolf, influence the optimal adaptive control rules and expected results. We introduce "RISK" to describe this influence. Compare equation (12). If RISK is 0, then we consider the system to be completely deterministic. If RISK=1, then we have the degree of risk that describes reality, according to the empirical data.

$$\begin{cases} \Delta M &= 0.372M - 0.178 \times 10^{-3}M^2 - 6.593W - u(\bullet) + M\varepsilon_M \times RISK \\ \Delta W &= -0.244W + 0.230 \times 10^{-3}MW - v(\bullet) + W\varepsilon_W \times RISK \end{cases} \tag{12}$$

Each year, the moose and wolf populations develop according to (12). The intervals between population control, in the form of hunting missions, are however five years. Several numerical specifications that are computationally necessary but have less general theoretical interest, are made in the software. The complete optimization model is included in the Appendix.

3. Results

The average optimal moose hunting level, Averu, is a decreasing function of the wolf population value parameter, P_{ARW}. This is found in Figure 1. If RISK = 0, we have Averu_0. The real degree of risk, RISK = 1, gives Averu_1. Hence, the average optimal moose hunting level is higher (Averu_1), a result of the real degree of risk (RISK = 1), than the result that we get if we assume that there is no risk (Averu_0). In Figure 2, we see that the optimal moose control limit, Optu0, which is the lowest level of the moose population that we accept without starting to hunt, is an increasing function of the wolf population value parameter, P_{ARW}. This is reasonable, since the wolf needs moose as food. If we want to keep a larger wolf population, we should hunt less intensively for moose. Optu0_0 and Optu0_1 represent the cases RISK = 0 and RISK = 1. The average optimal moose population level, AverM, increases more if the wolf population value parameter, P_{ARW} increases, under risk than under certainty. This is seen in Figure 3. This is reasonable. More wolfs need more moose as food. If we want to keep a larger wolf population, we should let the wolfs eat more moose. If the stochastic variations are larger, it is more important to have extra supplies of food available for a wolf population that sometimes is very large. AverM_0 and AverM_1 represent the cases RISK = 0 and RISK = 1.

Figure 4. shows that the optimal moose control limit, Optu0, which is the lowest level of the moose population that we accept without starting to hunt, is always lower than the average moose population, AverM. The graph shows the situation under risk. One important reason for the difference between the curves is that the hunting only occurs periodically. In the software, the default hunting interval is 5 years. In Figure 5., we see that, if RISK = 0, the optimal wolf control limit, Optv0_0, is very close to the average wolf population size, AverW_0. Of course, in reality, the stochastic variations in the population make this unrealistic. Compare Figure 6. In Figure 6., we see that, in reality, when RISK = 1, the optimal wolf control limit, Optv0_1, is much higher than the

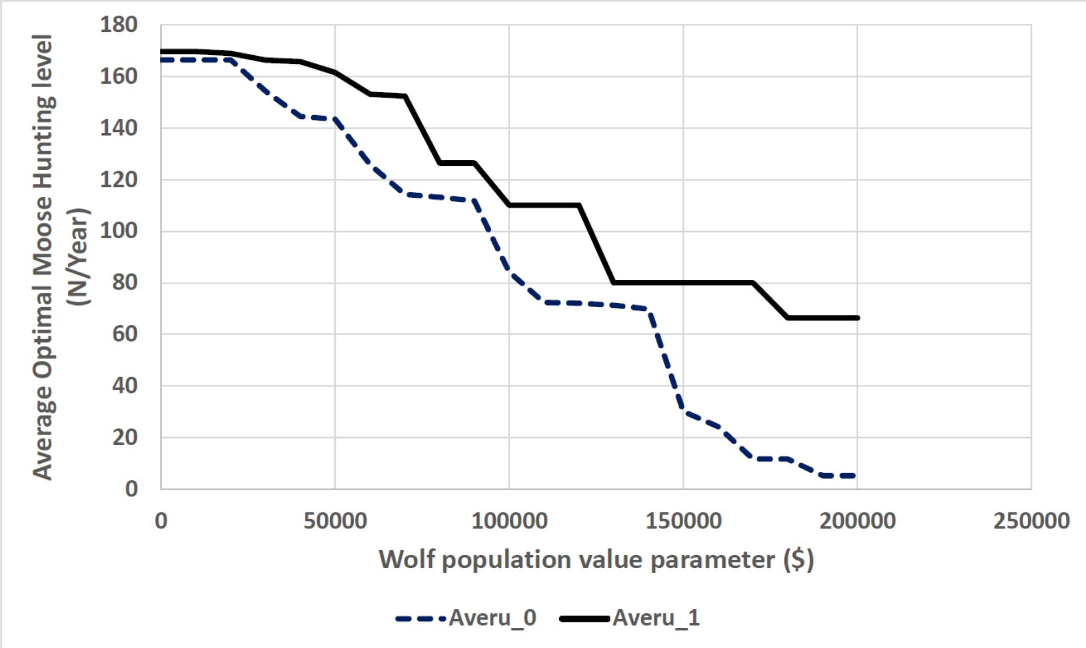

Figure 1.

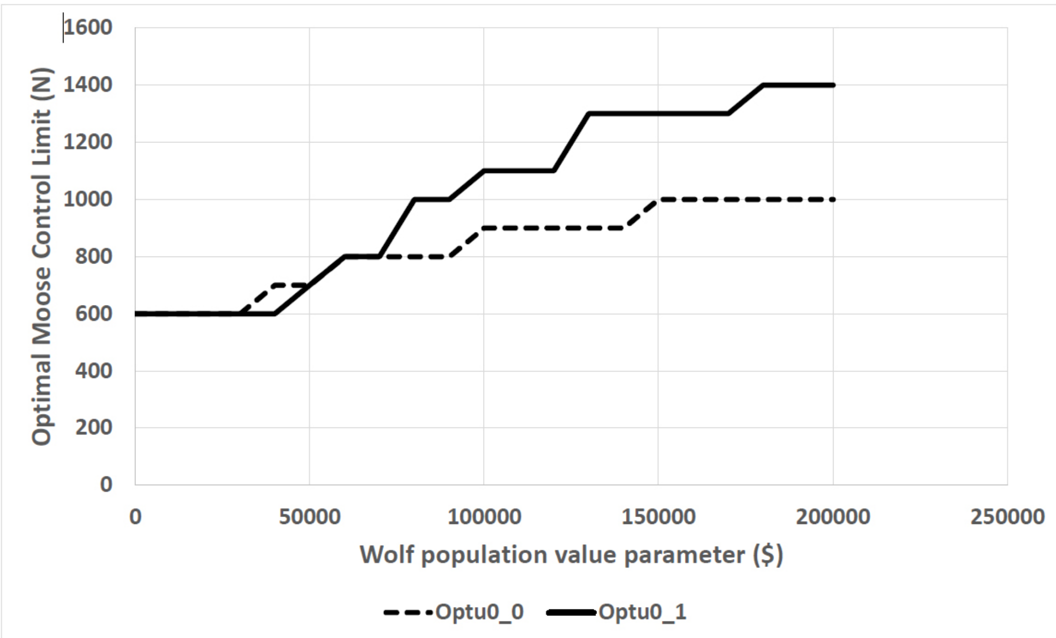

Figure 2.

average wolf population size, AverW_1. The stochastic variations in the population are considerable. With five year hunting intervals and because of the stochastic changes, the wolf population can grow far above the optimal wolf control limit, Optv0_1. If that happens, the hunt will reduce the population to the limit, Optv0_1. Stochastic changes

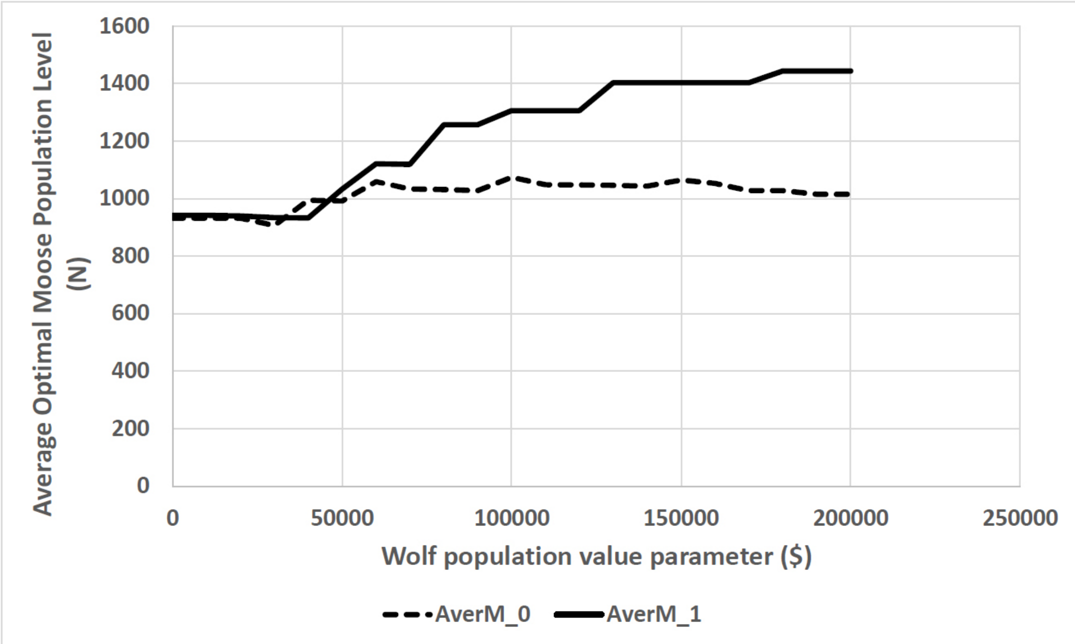

Figure 3.

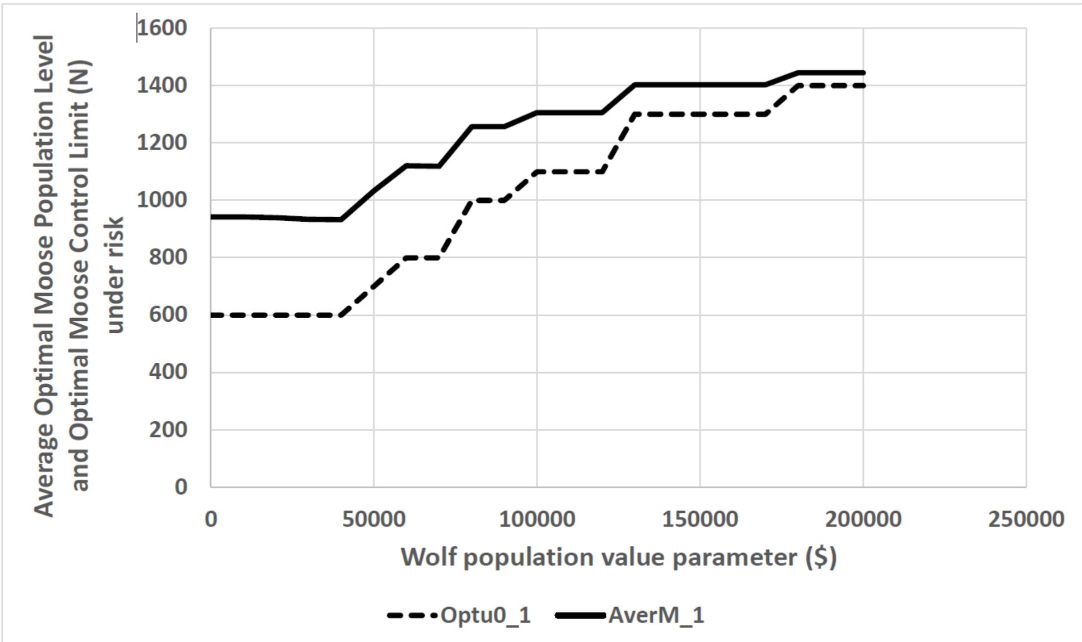

Figure 4.

can however also reduce the wolf population very much during five years. Then, no hunting takes place. For these reasons, with the realistic level of risk, RISK = 1, the average wolf population size should be expected to be far below the optimal wolf control limit, as we see in Figure 6. If the wolf population value parameter would be very high

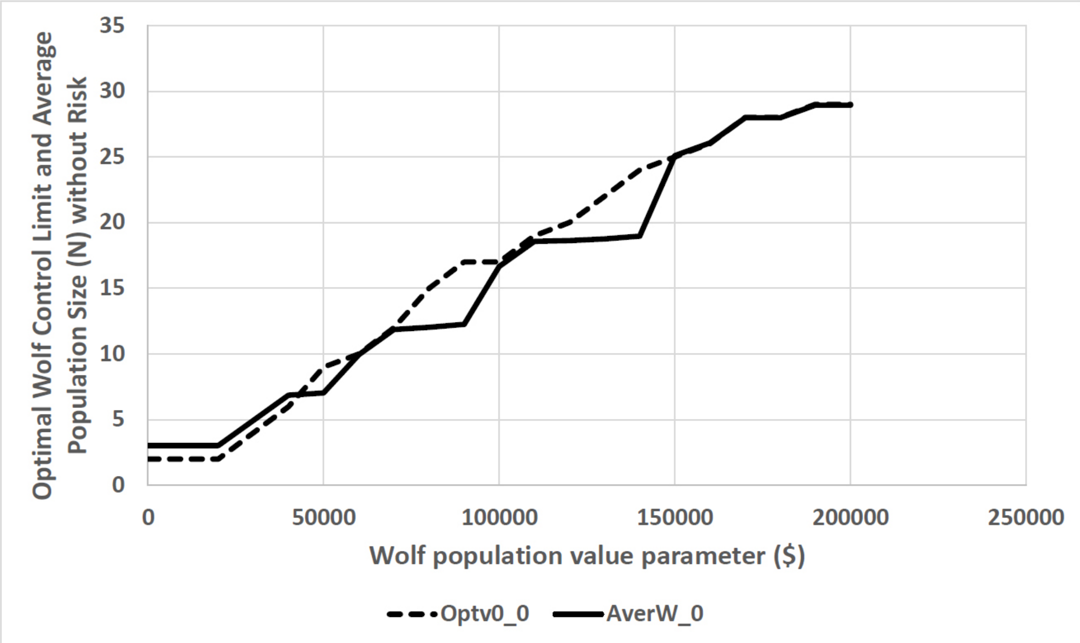

Figure 5.

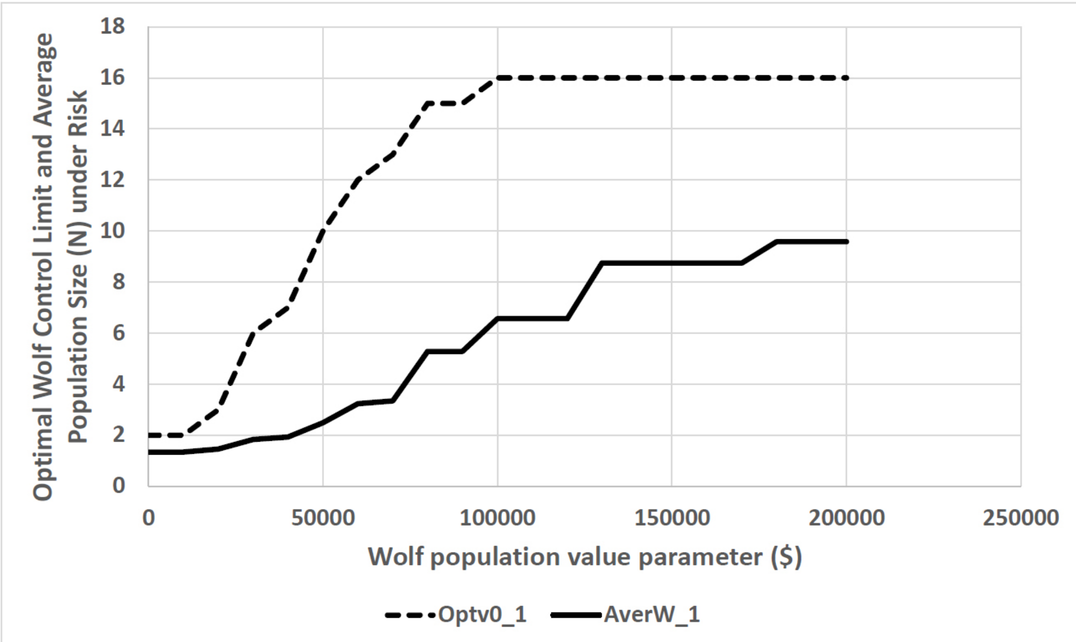

Figure 6.

and there would be no risk in the system, RISK = 0, we should focus on keeping the wolf population at a high and stable level. This is consistent with Figure 7. There, the average wolf hunting level, Averv, goes to zero. This is represented by Averv_0. If the risk is realistic, RISK = 1, however, and the wolf population is comparatively high, then there is a lot of stochastic variation in the wolf population. This is handled via a much higher average wolf hunting level. Compare Averv_1. In Figure 8., we see how the

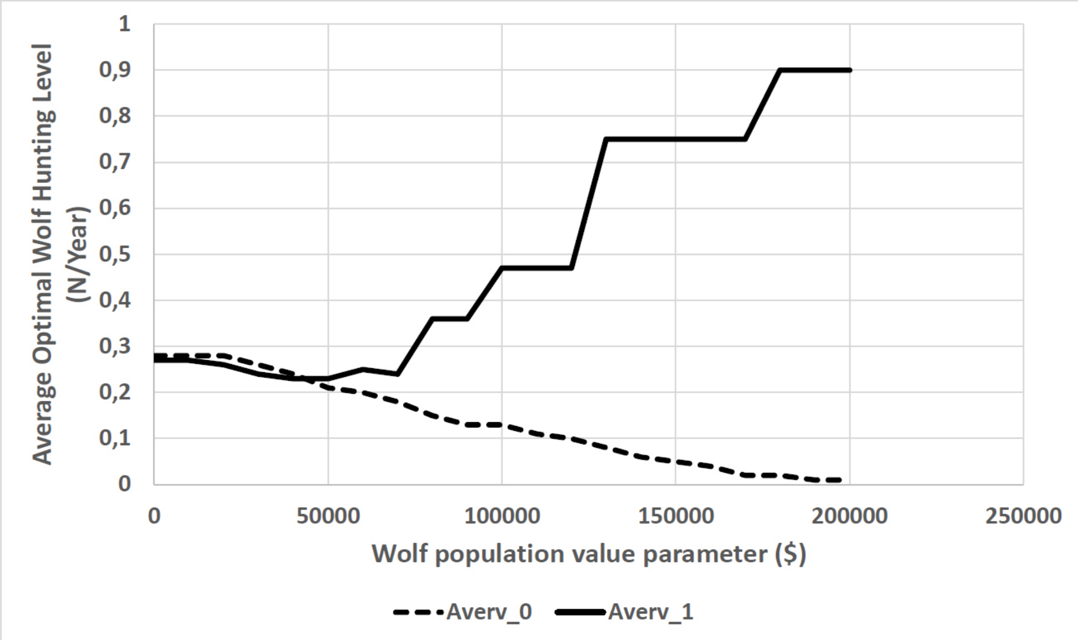

Figure 7.

average optimal wolf population increases with the wolf population value parameter and decreases with the level of risk. The alternatives AverW_0 and AverW_1 represent RISK = 0 and RISK = 1, respectively. These things are consistent with the results presented in Figures 1 – 7. Finally, as we see in Figure 9., the optimal objective function value, is a decreasing function of the level of risk in the stochastic system. MOptObj_0 and MOptObj_1 correspond to RISK = 0 and RISK = 1, respectively. This is quite reasonable. With stochastic variations, the population levels will randomly change from the levels that would be optimal with consideration of the objective function. The control efforts are optimized but they only occur periodically. Hence, they cannot continuously change the population levels. Furthermore, the controls only make it possible to reduce the population sizes. If the populations are "too low", it is not feasible in the model to "increase the population sizes". Of course, in reality, it is sometimes possible, to move some more wolfs to the area. This has been done in some countries to avoid extinction. This can also be a way to avoid genetic problems that can appear in small populations. In the optimization code, it would be possible to introduce options to increase the populations, simultaneously reducing the objective function with the relevant costs. Clearly, the objective function is an increasing and convex function of the wolf population value parameter. This follows from the fact that the number of wolves and the value per wolf simultaneously increase if the wolf population value parameter increases.

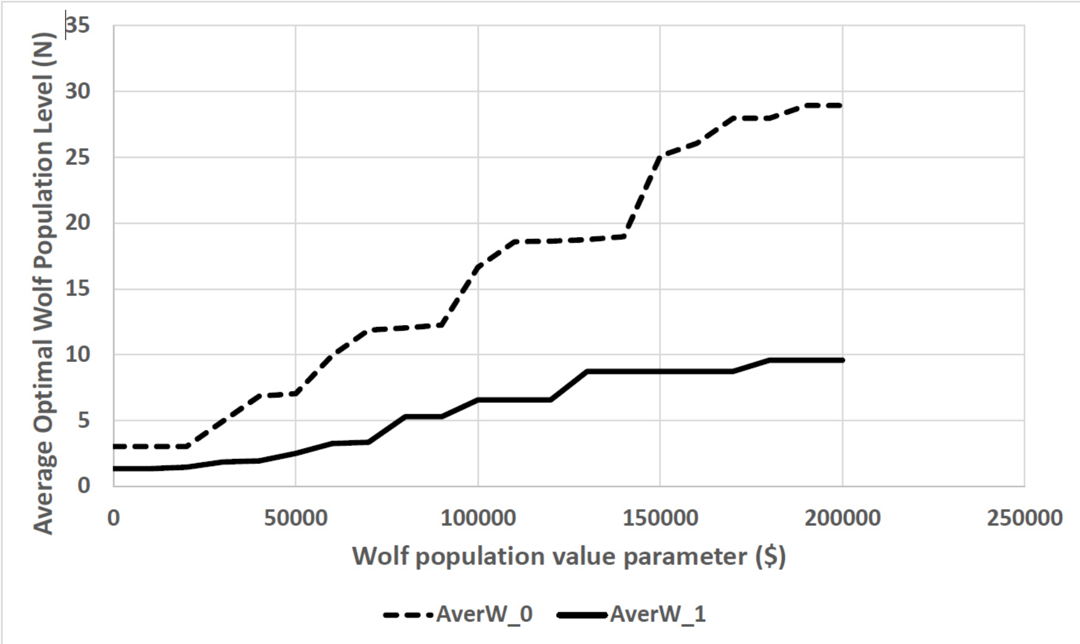

Figure 8.

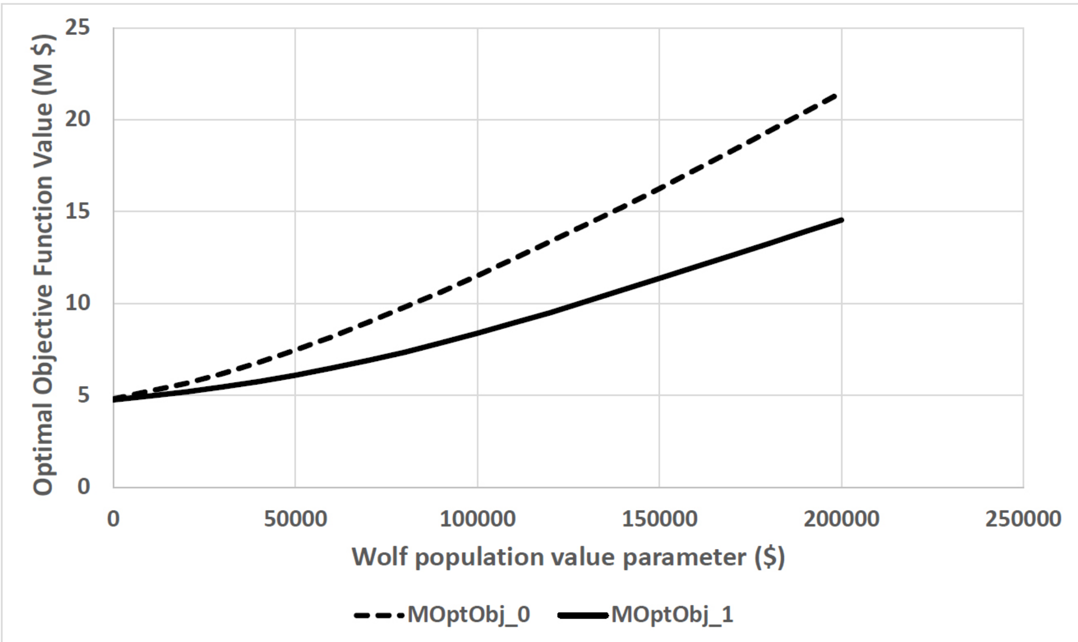

Figure 9.

4. Discussion

Would it have been possible to use analytical methods to solve the problems in this paper? The simple answer is: No. Fleming and Rishel (1975) present the theory of stochastic optimal control very well. However, it is quite clear that we can never expect to handle the relevant problem in this paper via stochastic optimal control theory in continuous time. Would it have been possible to optimize the management of the two species system with alternative methods? One method that is always available is stochastic dynamic programming. This approach is well described by Winston (2004). Then, however, it would have been necessary to estimate transition probability matrices to be used in the stochastic dynamic programming algorithm. Such transformations of the original problem introduce new approximations. The nonlinearities cause particular problems and most likely more errors appear. These efforts would not lead to a better solution of the original problem. With the method used in this manuscript, transformations and redefinitions are not necessary. You simply use the stochastic difference equations, as they are, in a repeated sequence. This is a more simple and direct method. Furthermore, if the number of stochastic scenarios in the method used in this paper, increases, the precision of the method also increases. The only problem is to select the desired level of precision, which is a question of processing time. The necessary processing time is however very limited in problems of this type. All of the results derived and presented in this paper, were derived with 98 seconds of CPU time with an Intel CORE i5, 8^{th} Generation, processor. Of course, it always takes time to develop software. On the other hand, the software is already available in the Appendix.

Now, let us consider the interests of stakeholders and policymakers, in the model and the results obtained in this study. First, we define a list of stakeholders or interest groups. Individual persons may belong to zero, one or several of these groups: The forest land owners, the forest products industry, the moose hunters, the general public, environmentally interested people, people that worry about global warming and the government.

The forest land owners are usually interested in economically profitable wood production. They prefer a stable population of moose, at a low and controllable level, since a moose population that is sometimes very large can destroy forest plantations completely. Hence, they benefit from a large and stable wolf population and/or a large and stable moose hunting level. They are expected to benefit from the controlled population levels that applications of the new model can lead to. The owners of forest products industry plants have interests in good and stable availability of timber of good quality. Large and unstable moose populations give more damages to timber quality and timber volume that smaller and more stable moose populations. A large and stable wolf population is good for them. An alternative is a large moose hunting level. Hence, they also benefit from population control based on applications of the new model.

The moose hunters benefit from a low wolf population and a comparatively large moose population. However, the moose hunters are also often forest land owners and may have interests in the forest products industries. In any case, the model is a tool that can be used to find the best compromise solution also for their different interests.

Environmentally interested people usually want to preserve stable wildlife populations of different kinds. Obviously, the wolf and moose populations have been very unstable during long periods without population control. Periodically, the wolf has been almost extinct. Furthermore, other animal and plant populations are influenced by wolves and moose and may also periodically suffer, without adaptive wolf and moose control. Hence, the new model should be useful also for these environmentally concerned groups.

The global temperature is an increasing function of CO_2 level in the atmosphere. Large and uncontrolled moose populations can destroy many trees that otherwise would absorb considerable amounts of CO_2 from the atmosphere. Hence, people interested in a cooler climate should be interested in a large and stable wolf population and/or a large level of stable moose hunting. The model is useful in such analyses. Finally, the general public and politicians of all kinds should be interested in the many possible applications of the model that the different interest groups should be interested in. Without population management based on analysis with an adaptive model of this type, the solutions cannot be assumed to become optimized for anybody.

5. Conclusions

A robust numerical approach to optimal adaptive integer pulse control of stochastic nonlinear systems has been created. This is important, because for most stochastic nonlinear systems, optimal adaptive control rules cannot be derived with analytical methods. The method is very general and can easily be applied to most adaptive nonlinear stochastic control problem, from engineering, technology, management or other fields.
The complete nonlinear adaptively controlled stochastic system is simulated during the period of interest, for a very large number of alternative sequences of stochastic disturbances, for every feasible integer combination of adaptive control rules. The optimal adaptive control rules that maximize the expected value of the objective function are selected as the optimal adaptive control rules.
The application showed, in several ways, that the level of risk in the nonlinear system is very important to the optimal adaptive control rules: a. The average optimal moose hunting level is a decreasing function of the wolf population value parameter and an increasing function of the level of risk in the predator prey system. b. The average optimal wolf population level is an increasing function of the wolf population value parameter and a decreasing function of the level of risk in the predator prey system.

Appendix A. The optimization model:

The following optimization program was used to calculate all results presented in this is paper. It is developed for QB64.

```
REM AdaptCtrl
REM Peter Lohmnader 210228
OPEN "C:\Users\Peter\Desktop\RESULTS\AdapOut.txt" FOR OUTPUT AS #1
DIM Epsilon(2, 100, 100), M(100, 100), W(100, 100), u(100, 100), v(100, 100),
regt(100), d(100)
NTOT = 100
FOR i = 1 TO 2
FOR j = 0 TO 100
FOR N = 1 TO NTOT
eps = 0
FOR k = 1 TO 12
eps = eps + RND
NEXT k
Epsilon(i, j, N) = eps - 6
NEXT N
```

```
NEXT j
NEXT i
REM Initial conditions
FOR N = 1 TO NTOT
M(0, N) = 1200
W(0, N) = 25
NEXT N
REM Discounting function
R = 0.03
FOR t = 0 TO 100
d(t) = EXP(-R * t)
NEXT t
PRINT " RISK PARW Optu0 Optv0 OptObj AverM AverW Averu Averv "
PRINT #1, " RISK PARW Optu0 Optv0 OptObj AverM AverW Averu Averv "
FOR RISK = 0 TO 1 STEP 1
STM = 0.174 * RISK
STW = 0.334 * RISK
FOR PARW = 0 TO 200000 STEP 10000
REM Default parameters of control rules
u0 = 1000
u1 = 1
v0 = 4
v1 = 1
timepath = 0
Interval = 5
REM Optimization of the objective function
OptObj = 0
FOR u0 = 500 TO 1500 STEP 100
FOR v0 = 2 TO 30
FOR N = 1 TO NTOT
FOR t = 1 TO 100
regt(t) = 0
test = INT(t / Interval) - t / Interval
IF test = 0 THEN regt(t) = 1
M(t, N) = M(t - 1, N) + 0.372 * M(t - 1, N) - 0.000178 * M(t - 1, N) * M(t - 1, N) -
6.593 * W(t - 1, N)
+ M(t - 1, N) * STM * Epsilon(1, t - 1, N)
W(t, N) = W(t - 1, N) - 0.244 * W(t - 1, N) + 0.000230 * M(t - 1, N) * W(t - 1, N)
+ W(t - 1, N) * STW * Epsilon(2, t - 1, N)
M(t, N) = INT(M(t, N) + 0.5)
W(t, N) = INT(W(t, N) + 0.5)
u(t, N) = 0
v(t, N) = 0
IF regt(t) = 0 THEN GOTO 100
uev = M(t, N) - u0
IF uev > 0 THEN u(t, N) = INT(uev * u1 + 0.5)
IF u(t, N) > M(t, N) THEN u(t, N) = M(t, N)
vev = W(t, N) - v0
IF vev > 0 THEN v(t, N) = INT(vev * v1 + 0.5)
IF v(t, N) > W(t, N) THEN v(t, N) = W(t, N)
```

```
M(t, N) = M(t, N) - u(t, N)
W(t, N) = W(t, N) - v(t, N)
100 REM
IF M(t, N) < 0 THEN M(t, N) = 0
IF W(t, N) < 0 THEN W(t, N) = 0
NEXT t
IF timepath = 0 THEN GOTO 200
PRINT #1, ""
PRINT #1, " t M W regt u v"
FOR t = 0 TO 100
PRINT USING "########"; t; M(t, N); W(t, N); regt(t); u(t, N); v(t, N)
PRINT #1, USING "#####.##"; t; M(t, N); W(t, N); regt(t); u(t, N); v(t, N)
NEXT t
200 REM
NEXT N
obj = 0
FOR N = 1 TO NTOT
FOR t = 1 TO 100
obj = obj + d(t) * (1000 * u(t, N) + 1000 * v(t, N) + PARW * LOG(1 + W(t, N)))
NEXT t
NEXT N
obj = obj / NTOT
IF obj < OptObj THEN GOTO 300
Optu0 = u0
Optv0 = v0
OptObj = obj
AverM = 0
AverW = 0
Averu = 0
Averv = 0
FOR N = 1 TO NTOT
FOR t = 1 TO 100
AverM = AverM + M(t, N) / NTOT / 100
AverW = AverW + W(t, N) / NTOT / 100
Averu = Averu + u(t, N) / NTOT / 100
Averv = Averv + v(t, N) / NTOT / 100
NEXT t
NEXT N
300 REM
NEXT v0
NEXT u0
PRINT USING "#######.##"; RISK;
PRINT USING "##########"; PARW; Optu0; Optv0; OptObj;
PRINT USING "#######.##"; AverM; AverW; Averu; Averv
PRINT #1, USING "#######.##"; RISK;
PRINT #1, USING "##########"; PARW; Optu0; Optv0; OptObj;
PRINT #1, USING "#######.##"; AverM; AverW; Averu; Averv
NEXT PARW
NEXT RISK
END
```

Appendix B. Table

Table B1. Optimized results from the optimization model:

RISK	PARW	Optu0	Optv0	OptObj	AverM	AverW	Averu	Averv
0.00	0	600	2	4800935	932.75	3.02	166.46	0.28
0.00	10000	600	2	5227138	932.75	3.02	166.46	0.28
0.00	20000	600	2	5653461	932.75	3.02	166.46	0.28
0.00	30000	600	4	6178636	906.71	4.94	154.30	0.26
0.00	40000	700	6	6797018	994.93	6.86	144.56	0.24
0.00	50000	700	9	7462465	991.99	7.04	143.43	0.21
0.00	60000	800	10	8181700	1059.05	9.94	125.89	0.20
0.00	70000	800	12	8981711	1033.98	11.86	114.30	0.18
0.00	80000	800	15	9796166	1031.48	12.03	113.24	0.15
0.00	90000	800	17	10624063	1028.33	12.25	111.85	0.13
0.00	100000	900	17	11504882	1074.29	16.66	84.23	0.13
0.00	110000	900	19	12431596	1048.98	18.58	72.45	0.11
0.00	120000	900	20	13368959	1048.33	18.63	72.14	0.10
0.00	130000	900	22	14310103	1046.53	18.75	71.36	0.08
0.00	140000	900	24	15259542	1043.34	18.97	69.91	0.06
0.00	150000	1000	25	16247772	1065.17	25.10	30.22	0.05
0.00	160000	1000	26	17277430	1053.37	26.06	24.33	0.04
0.00	170000	1000	28	18309208	1027.94	27.98	11.79	0.02
0.00	180000	1000	28	19359250	1027.94	27.98	11.79	0.02
0.00	190000	1000	29	20414604	1015.34	28.94	5.33	0.01
0.00	200000	1000	29	21474132	1015.34	28.94	5.33	0.01
1.00	0	600	2	4751509	942.09	1.34	169.73	0.27
1.00	10000	600	2	4961350	942.09	1.34	169.73	0.27
1.00	20000	600	3	5180255	940.21	1.46	168.92	0.26
1.00	30000	600	6	5450907	934.41	1.84	166.41	0.24
1.00	40000	600	7	5749353	932.89	1.93	165.76	0.23
1.00	50000	700	10	6090864	1033.09	2.49	161.59	0.23
1.00	60000	800	12	6476809	1121.58	3.23	153.21	0.25
1.00	70000	800	13	6895000	1118.93	3.34	152.45	0.24
1.00	80000	1000	15	7335542	1257.38	5.28	126.54	0.36
1.00	90000	1000	15	7851468	1257.38	5.28	126.54	0.36
1.00	100000	1100	16	8376443	1305.87	6.57	110.15	0.47
1.00	110000	1100	16	8937077	1305.87	6.57	110.15	0.47
1.00	120000	1100	16	9497668	1305.87	6.57	110.15	0.47
1.00	130000	1300	16	10121377	1403.28	8.74	80.13	0.75
1.00	140000	1300	16	10745463	1403.28	8.74	80.13	0.75
1.00	150000	1300	16	11369613	1403.28	8.74	80.13	0.75
1.00	160000	1300	16	11993697	1403.28	8.74	80.13	0.75
1.00	170000	1300	16	12617788	1403.28	8.74	80.13	0.75
1.00	180000	1400	16	13252275	1444.42	9.58	66.43	0.90
1.00	190000	1400	16	13896613	1444.42	9.58	66.43	0.90
1.00	200000	1400	16	14540942	1444.42	9.58	66.43	0.90

References

[1] Blanchard, P., Devaney, R.L., Hall, G.R., 2006. Differential Equations, 3^{rd} Ed., Thomson Brooks/Cole, USA, 828 p.

[2] Braun, M., 1983. Differential Equations and Their Applications, 3^{rd} Ed., Springer-Verlag, New York, 546 p.

[3] Fleming, W.H., Rishel, R.W., 1975, Deterministic and Stochastic Optimal Control, Springer-Verlag, New York, 222 p.

[4] Fryxell, J.M., Falls, J., Falls, A., Brooks, R.J., Dix, L., Strickland, M.A., 1999, Density dependence, prey dependence, and population dynamics of martens in Ontario, Ecology, 80(**??**), 1311-1321.

[5] Khiabani, A.G., 2020. Application of Optimal Switching Using Adaptive Dynamic Programming in Power Electronics, Mechanical Engineering Research Theses and Dissertations. 25, Southern Methodist University, Dallas, Texas, USA, 92 p.

[6] Knadler, C. E., 2008. Models of a predator-prey relationship in a closed habitat. 2008 Winter Simulation Conference. https://doi.org/10.1109/WSC.2008.4736407

[7] Lohmander, P., 2017. Optimal Stochastic Control in Continuous Time with Wiener Processes: General Results and Applications to Optimal Wildlife Management, Iranian Journal of Operations Research, Vol. 8, No. 2, 58-67.

[8] Lotka A.J.,1920. Analytical Note on Certain Rhythmic Relations in Organic Systems. Proc Natl Acad Sci USA. 6(**??**):410-415.

[9] Michigan Tech, 2021. Wolves and Moose of Isle Royale, About the project. https://www.isleroyalewolf.org/overview/overview/at_a_glance.html

[10] Navabi, M., Mirzaei, H., 2017. Robust Optimal Adaptive Trajectory Tracking Control of Quadrotor Helicopter. Latin American Journal of Solids and Structures [online]. 2017, v. 14, n. 6 [Accessed 9 June 2021] , pp. 1040-1063.

[11] Piga, D., Bemporad, A., 2020. New trends in modeling and control of hybrid systems, International Journal of Robust Nonlinear Control, 30: 5775– 5776.

[12] Saleem, O., Rizwan, M., Mahmood-ul-Hasan, K:, Ahmad, M. 2020. Performance enhancement of multivariable model reference optimal adaptive motor speed controller using error-dependent hyperbolic gain functions, Automatika, 61(**??**), 117-131.

[13] Sethi, S.P., Thompson, G.L., 2000. Optimal Control Theory, Applications to Management Science and Economics, 2^{nd} ed., Kluwer Academic Publishers, USA, 504 p.

[14] Tung, K.K., 2007. Topics in Mathematical Modeling, Princeton University Press, Princeton University Press, New Jersey, 300 p.

[15] Turri, V., Flardh, O., Martensson, J., Johansson, K.H., 2018. Fuel-optimal look-ahead adaptive cruise control for heavy-duty vehicles, Annual American Control Conference (ACC) June 27–29, Wisconsin Center, Milwaukee, USA, 1841-1848.

[16] Winston, W.L., 2004. Introduction to probability models, Operations Research: Volume two, Thomson, Brooks/Cole, Belmont, CA, USA, 729 p.

[17] Yeo, S.H., Aligiri, E., Tan, P.C., Zarepour, H., 2009. An Adaptive Speed Control System for Micro Electro Discharge Machining, CPl 181, Third Manufacturing Engineering Society International Conference, Segui V.J. and Reig M.J. (Editors), American Institute of Physics, Melville, NY, USA, 61-72.